FINAL EXAM | ISI BANGALORE | NOVEMBER 2014

ALGEBRA III | 100 POINTS

Notes.
(a) Begin each answer on a separate sheet and ensure that the answers to all the parts to a question are arranged contiguously.
(b) Assume only those results that have been proved in class. All other steps should be justified.
(c) $\mathbb{Z}$ = integers, $\mathbb{Q}$ = rational numbers, $\mathbb{R}$ = real numbers $\mathbb{C}$ = complex numbers.

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1. [10 points] Find an $n \times n$ matrix over $\mathbb{Q}$ whose characteristic polynomial is
$$\sum_{i=1}^{n+1} iX^{n-i+1} = X^n + 2X^{n-1} + \cdots + (n+1).$$

2. [12 points] Let G be the abelian group generated by x, y, z satisfying the relations
$$3x + 2y + 8z = 0, \qquad 2x + 4z = 0.$$
Write G as a direct product of cyclic groups.

3. [20 points] Which of the following are algebraic integers ? (Give brief justifications for your answers.)
$$\sqrt{2} + 1, \quad 2(2^{1/3}) - 1, \quad (\sqrt{3} + 1)/2, \quad (\sqrt{5} + 1)/2.$$

4. [16 points]
Using factorization in the ring of Gaussian integers, find all the different ways of writing $1105 = 5 \cdot 13 \cdot 17$ as a sum of two squares in $\mathbb{Z}$. DO NOT give an answer via a brute force calculation of squares in $\mathbb{Z}$.

5. [6 points] Factor $7 + i$ as a product of primes in $\mathbb{Z}[i]$.

6. [15 points] Suppose $f(x)$ is a monic polynomial of degree 3 in $\mathbb{Z}[X]$ and suppose there exists a prime p in $\mathbb{Z}$ such that p does not divide the product $f(0) \cdot f(1) \cdots f(p-1)$. Prove that f is irreducible over $\mathbb{Q}$.

7. [6 points] Prove that there are infinitely many units in the ring $\mathbb{Z}[\sqrt{3}]$. (Hint: $2 + \sqrt{3}$ is a unit.)

8. [15 points] Let L be a non-constant linear polynomial in $\mathbb{R}[X, Y]$. Prove that $(X^2 + Y^2 + 1, L)$ is a maximal ideal in $\mathbb{R}[X, Y]$.